

DIMENSIONAL REGULARIZATION OF THE GRAVITATIONAL INTERACTION OF POINT MASSES IN THE ADM FORMALISM*

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The ADM formalism for two-point-mass systems in d space dimensions is sketched. It is pointed out that the regularization ambiguities of the 3rd post-Newtonian ADM Hamiltonian considered directly in $d = 3$ space dimensions can be cured by dimensional continuation (to complex d 's), which leads to a finite and unique Hamiltonian as $d \rightarrow 3$. Some so far unpublished details of the dimensional-continuation computation of the 3rd post-Newtonian two-point-mass ADM Hamiltonian are presented.

Keywords: binary systems, equations of motion, point masses, dimensional regularization

1. Introduction

The problem of finding the equations of motion (EOM) of a two-body system within the post-Newtonian (PN) approximation of general relativity is solved up to the 3.5PN order of approximation for the case of compact and nonrotating bodies [by n PN approximation we mean corrections of order $(v/c)^{2n} \sim (Gm/(rc^2))^n$ to Newtonian gravity]. The 3PN level of accuracy was achieved only recently. There exist two independent derivations of the 3PN EOM using distributional (Dirac delta's) sources: either ADM-Hamiltonian-based,^{1,2} or harmonic-coordinate-based.^{3,4} There also exists a third independent derivation of the 3PN EOM in harmonic coordinates using a surface-integral approach.⁵

To cure the self-field divergencies of point particles it is necessary to use some regularization method. It turned out that different such methods applied in $d = 3$ space dimensions were not able to give unique EOM at the 3PN order. Only by em-

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ploying dimensional continuation was it possible to obtain unambiguous results.^{2,4} In this note we review the dimensional-continuation-based derivation of the 3PN two-point-mass ADM Hamiltonian.

2. ADM formalism for 2-point-mass systems in d space dimensions

We use units such that $c=16\pi G_{d+1}=1$. We work in an asymptotically flat $(d+1)$ -dimensional spacetime with Minkowskian coordinates x^0 , $\mathbf{x} \equiv (x^1, \dots, x^d)$. Particles are labeled by the index $a \in \{1, 2\}$; masses, positions, and momenta of the particles are denoted by m_a , $\mathbf{x}_a \equiv (x_a^1, \dots, x_a^d)$, and $\mathbf{p}_a \equiv (p_{a1}, \dots, p_{ad})$, respectively. We also define: $\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a$, $r_a := |\mathbf{r}_a|$, $\mathbf{n}_a := \mathbf{r}_a/r_a$; $\mathbf{r}_{12} := \mathbf{x}_1 - \mathbf{x}_2$, $r_{12} := |\mathbf{r}_{12}|$ ($|\mathbf{v}|$ means here the Euclidean length of the d -vector \mathbf{v}). The canonical variables of the theory consist of *matter* variables $(\mathbf{x}_a, \mathbf{p}_a)$ and *field* variables (γ_{ij}, π^{ij}) , where the space metric γ_{ij} is induced by the full space-time metric on the hypersurface $x^0 = \text{const}$; its conjugate π^{ij} can be expressed in terms of the extrinsic curvature of that hypersurface.

Source terms in the *constraint equations* written down for two-point-mass systems are proportional to the d -dimensional Dirac delta functions $\delta(\mathbf{x} - \mathbf{x}_a)$. We use the ADM gauge defined by the conditions (TT \equiv transverse-traceless):

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)} \delta_{ij} + h_{ij}^{\text{TT}}, \quad \pi^{ii} = 0. \quad (1)$$

The field momentum π^{ij} splits into a TT part π_{TT}^{ij} and a rest $\tilde{\pi}^{ij}$ (traceless but expressible in terms of a vector), $\pi^{ij} = \tilde{\pi}^{ij} + \pi_{\text{TT}}^{ij}$. If both the constraint equations and the gauge conditions are satisfied, the ADM Hamiltonian can be put into its *reduced* form:

$$H(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}) = - \int d^d x \Delta \phi(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}). \quad (2)$$

The PN expansion of the reduced Hamiltonian is worked out up to the 3.5PN order:

$$H = \sum_{a=1}^2 m_a + H_N + H_{1\text{PN}} + H_{2\text{PN}} + H_{2.5\text{PN}} + H_{3\text{PN}} + H_{3.5\text{PN}} + \mathcal{O}((v/c)^8). \quad (3)$$

3. Dimensional regularization of the 3PN Hamiltonian

In Refs. 1 it was shown that the Riesz-implemented Hadamard regularization of the 3PN two-point-mass Hamiltonian performed in $d = 3$ space dimensions gives ambiguous results. The ambiguities were parametrized by two numerical coefficients called ambiguity parameters and denoted by ω_{kinetic} and ω_{static} .

Dimensional continuation consists in obtaining the 3-dimensional Hamiltonian as $\lim_{d \rightarrow 3} H_{3\text{PN}}(d)$, where $H_{3\text{PN}}(d)$ is the Hamiltonian computed in d space dimensions. This can be done straightforwardly if no poles proportional to $1/(d-3)$ arise when $d \rightarrow 3$ (or if one shows that these poles can be renormalized away, as happens

in harmonic coordinates⁴). Reference 2 has shown that out of all terms building up the Hamiltonian density there are ten terms $T_A(d)$, $A = 1, \dots, 10$, giving rise to poles when $d \rightarrow 3$. It was checked that the poles produced by these terms cancel each other, thus $\lim_{d \rightarrow 3} H_{3\text{PN}}(d)$ exists. Moreover, it was shown that for all other terms the 3-dimensional regularization give the same results as dimensional continuation.

Let $H_{3\text{PN}}^{\text{Had}}$ be the 3PN Hamiltonian obtained in Refs. 1 by using an Hadamard “partie finie” (Pf) regularization defined in $d = 3$ space dimensions. To correct this Hamiltonian one needs to compute the difference $\Delta H_{3\text{PN}} := \lim_{d \rightarrow 3} H_{3\text{PN}}(d) - H_{3\text{PN}}^{\text{Had}}$. Only ten terms T_A contribute to $\Delta H_{3\text{PN}}$, therefore

$$\Delta H_{3\text{PN}} = \lim_{d \rightarrow 3} \int d^d x \sum_{A=1}^{10} T_A(d) - \text{Pf} \int d^3 x \sum_{A=1}^{10} T_A(3). \quad (4)$$

Below we present three different methods which we used to compute $\Delta H_{3\text{PN}}$. The details of the 2nd and 3rd method were not published so far. Knowing $\Delta H_{3\text{PN}}$ one determines the values of both ambiguity parameters: $\omega_{\text{kinetic}} = 41/24$, $\omega_{\text{static}} = 0$.

1st method. In Ref. 2 $\Delta H_{3\text{PN}}$ was computed by means of the analysis of the local behaviour of the terms T_A around the particle positions $\mathbf{x} = \mathbf{x}_a$.

2nd method. It is possible to compute all d -dimensional integrals in Eq. (4) explicitly. To do this one uses the Riesz formula

$$\int d^d x r_1^\alpha r_2^\beta = \pi^{d/2} \frac{\Gamma((\alpha + d)/2) \Gamma((\beta + d)/2) \Gamma(-(\alpha + \beta + d)/2)}{\Gamma(-\alpha/2) \Gamma(-\beta/2) \Gamma((\alpha + \beta + 2d)/2)} r_{12}^{\alpha + \beta + d}, \quad (5)$$

and the distributional differentiation of homogeneous functions, e.g.,

$$\frac{\partial^2}{\partial x^i \partial x^j} \frac{1}{r_a^{d-2}} = \text{Pf} \left((d-2) \frac{d n_a^i n_a^j - \delta_{ij}}{r_a^d} \right) - \frac{4\pi^{d/2}}{d \Gamma(d/2 - 1)} \delta_{ij} \delta(\mathbf{x} - \mathbf{x}_a). \quad (6)$$

3rd method. Instead of d -dimensional Dirac distributions δ one uses d -dimensional Riesz kernels δ_{ε_a} to model point particles:

$$\delta(\mathbf{x} - \mathbf{x}_a) = \lim_{\varepsilon_a \rightarrow 0} \delta_{\varepsilon_a}(\mathbf{x} - \mathbf{x}_a), \quad \delta_{\varepsilon_a}(\mathbf{x} - \mathbf{x}_a) := \frac{\Gamma((d - \varepsilon_a)/2)}{\pi^{d/2} 2^{\varepsilon_a} \Gamma(\varepsilon_a/2)} r_a^{\varepsilon_a - d}. \quad (7)$$

Then one uses the formula (5) to calculate the integrals in Eq. (4) and, at the end of the calculation, one takes the limit $\varepsilon_1 \rightarrow 0$, $\varepsilon_2 \rightarrow 0$. No distributional differentiation is needed.

We have shown that these three methods yield the same final results.

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